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Renormalisation Group Theory of Branching Potts Interfaces

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Abstract

We develop a field-theoretic representation for the configurations of an interface between two ordered phases of a q -state Potts model in two dimensions, in the solid-on-solid approximation. The model resembles the field theory of directed percolation and may be analysed using similar renormalisation group methods. In the one-loop approximation these reveal a simple mechanism for the emergence of a critical value q_c , such that for $q < q_c$ the interface becomes a fractal with a vanishing interfacial tension at the critical point, while for $q > q_c$ the interfacial width diverges at a finite value of the tension, indicating a first-order transition. The value of the Widom exponent for $q < q_c$ within this approximation is in fair agreement with known exact values. Some comments are made on the case of quenched randomness. We also show that the $q \rightarrow -\infty$ limit of our model corresponds to directed percolation and that the values for the exponents in the one-loop approximation are in reasonable agreement with accepted values.

1 Introduction

The study of interfaces between ordered phases, besides being of interest for its own sake, often also provides useful information on the bulk properties of the system in question, especially close to a bulk critical point. For example, at a continuous transition, the interfacial tension is supposed to vanish as $(T_c - T)^\mu$, where the Widom exponent μ is related to the conventional bulk correlation length exponent by the scaling law $\mu = (D - 1)\nu$, where D is the overall dimensionality of the system [2].

When there are only two coexisting phases, as in a ferromagnetic Ising model, the structure of such an interface is relatively simple, especially at low temperatures. An interface in a two-dimensional system¹ may be generated by imposing suitable conditions on the Ising-like degrees of freedom $s(t, x) = \pm 1$ at the boundary of a finite but large box ($0 \leq t \leq L, |x| \leq L/2$): for example that $s = 1$ on $x = L/2, (0, x > 0)$ and $(L, x > 0)$, while $s = -1$ on $x = -L/2, (0, x < 0)$ and $(L, x < 0)$. At low temperatures the resultant interface has the form of a *directed path* from $(0, 0)$ to $(L, 0)$, that is, there are no overhangs. In addition, at low temperatures, bubbles of the wrong phase in the bulk regions above and below the interface are suppressed. The approximation of allowing only such directed paths gives the *solid-on-solid* (SOS) model for the interface. It is extremely easy to analyse, for example by transfer matrix methods, because the directed path may be thought of as a discrete time version of a simple random walk in the x -direction, with t playing the role of time. Alternatively, the partition function may be viewed as the imaginary time version of the Feynman path integral for a non-relativistic particle. Although it is strictly valid only at low temperatures, the SOS model does in fact capture accurately several important aspects of the true critical behaviour as $T \rightarrow T_c^-$. In particular it yields the exact

¹We use coordinates t and x parallel and perpendicular to the interface, for reasons which will become clear.

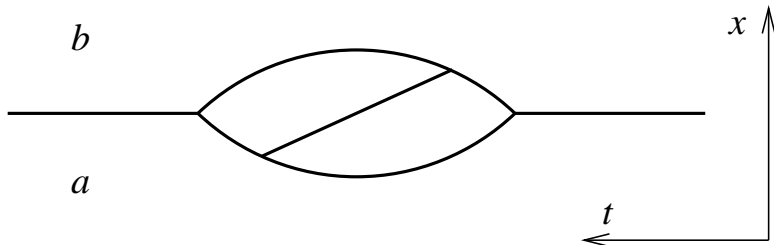


Figure 1: Typical configuration of a branching interface separating ordered phases a and b . This particular set of contributions has weight $(q-2)(q-3)$.

value $\mu = 1$ for the Ising model. It is understood that this success arises from a cancellation between overhang and bubble contributions in certain quantities which is probably peculiar to the Ising model. Nevertheless the ease of analysis suggests that it might be useful to extend it to other systems.

When there are more than two coexisting phases, as in general for the q -state Potts model, the structure of the interface is more complicated, since it can branch as illustrated in Fig. 1. At low temperatures, such branching should be suppressed since it locally increases the interfacial tension, but near the critical point this can be offset by entropic gains. Thus it is necessary to take into account all possible branchings near T_c . In general this is very difficult, but once again within the SOS approximation progress is possible. This is because, in the random walk analogy, one now has to consider random walks which may branch and coalesce. Similar processes occur in the field-theoretic formulation of directed percolation (DP) [3], which represents a rather generic class of non-equilibrium phase transitions. Examples are the so-called branching and annihilating random walks (BARW) [4] which have recently been studied rather extensively. Using the Feynman path integral analogy, or by other methods, it is possible to set up a field theory describing the diffusion, branching and annihilation processes of such many-particle systems. This is then amenable to standard methods of renormalisation group analysis.

In fact, branching interfaces within the SOS approximation have a number of important differences from ordinary interacting random walks. If Fig. 1 is simply viewed as a Feynman diagram in configuration space, a given propagator joining two vertices represents a sum over *all* directed paths between the vertices, including paths which may cross those corresponding to other lines in the diagram. However, it does not make physical sense for interfaces to cross. This may be taken into account by introducing an infinitely strong local repulsion between the particles in the field theory. In one transverse x dimension, this is equivalent to assuming that the particles behave as fermions, at least in between the branching and coagulation events. This turns out to be the simplest approach from a calculational point of view.

The other important difference is in the weighting factors. A given set of interfacial configurations such as that in Fig. 1 should be weighted by the number of ways of colouring the diagram with the q colours of the Potts model, consistent with the requirement that no two colours on either side of a given segment of interface are the same, and that the two (different) colours above and below the whole interface are fixed. A given diagram with n branchings (and n annihilations) will thus have a weight which is a polynomial in q of degree n . No such weighting factors arise for DP or ordinary BARWs, and it is not *a priori* obvious that they will lead to a diagrammatic structure which satisfies the correct Schwinger-Dyson equations so as to be multiplicatively renormalisable. However, in the next Section we argue that these diagrams can in fact be derived from a local lagrangian and therefore should be amenable to conventional renormalisation methods.

One of the well-known features of the bulk q -state Potts model is that there exists a critical value q_c such that, for $q \leq q_c$, the transition is continuous, with a diverging correlation length, while for $q > q_c$ it is first-order. In $D = 2$ dimensions $q_c = 4$ [5]. The physics behind this change of behaviour is not completely understood. For very large q it is possible to see from the mapping to the random cluster model that at the critical point there

are two states (either all sites are in their own separate cluster, or all sites are in the same infinite cluster) which have the same free energy but very different internal energies, corresponding to first-order coexistence. The inclusion of fluctuation effects does not alter this conclusion to any finite order in an expansion in powers of $1/q$. Clearly this cannot persist to $q = 2$ (the Ising model) when the transition is continuous. However, in mean field theory $q_c = 2$, and studies within the ϵ -expansion below six dimensions indicate that this value persists, at least to $D = 4$. On the other hand, simple real space renormalisation group calculations tend to predict the transition to be continuous for all finite q . It is only when additional lattice degrees of freedom corresponding to vacancies are introduced that a scenario is found which allows the emergence of a non-trivial value of q_c [6].

From the point of view of the interface, this change in behaviour as a function of q should be signalled by a vanishing renormalised interfacial tension at the transition for $q < q_c$, and a finite value for $q > q_c$. This is indeed what we find within the SOS approximation described above. Apart from q , the continuum version of our model contains two parameters: the bare interfacial tension σ_0 and the Boltzmann weight u_0 for a branching (or coagulation). On a given lattice these are of course smooth functions of the reduced coupling J/kT of the Potts model, but they turn out to renormalise differently. Our renormalisation group equations are based on the way that the dimensionless renormalised branching rate g (which, roughly speaking, measures the probability of a large-scale branching of the interface into two separate parts) varies with the renormalised interfacial tension σ , keeping the bare branching weight u_0 fixed. This equation has the form

$$\sigma \frac{\partial g}{\partial \sigma} \equiv \beta(g) = -\frac{1}{4}g - b(q - q_0)g^3 + O(g^5), \quad (1)$$

where the right hand side is the result of a one-loop calculation, with b a positive constant, and $q_0 = \frac{24}{5}$. Within this approximation we already see interesting behaviour. As σ increases (corresponding to $T \rightarrow 0$), $g \rightarrow$

0 as expected. On the other hand, as $\sigma \rightarrow 0$, for $q < q_0$ we see that g approaches a finite fixed point value, indicating that at the critical point there is branching on all scales and the interface is a fractal. For $q > q_0$, on the other hand, integration of (1) shows that, starting from a finite value for some large low-temperature value of σ_0 , g actually diverges at a *finite* value of σ . Although of course this is outside the region of validity of the one-loop approximation, it does nevertheless provide a mechanism for the existence of a lower bound on the renormalised interfacial tension, indicating that the transition must be first-order. Once g diverges, the interface will break up into many components and the bulk ordered phases will no longer be distinct. We may therefore identify this point with the critical temperature, and q_0 with the critical value q_c , in this approximation.

A more detailed analysis for $q < q_0$ allows the value of the Widom exponent to be extracted. Within the same approximation, we find for example $\mu \approx \frac{9}{11}$ for $q = 3$, to be compared with the exact value of $\frac{5}{6}$.

As has been remarked elsewhere [1], the configurations of the interface resemble those of the diagrams in the perturbative expansion for the connectness function in directed percolation (DP). Indeed, it is straightforward to show that taking the formal limit $q \rightarrow -\infty$ with $u_0^2|q|$ fixed in our diagrams give precisely the correct weights for DP. We may therefore compare our one-loop results for the exponents with those of DP, which are known very accurately in one transverse dimension. We find $\nu_{\parallel} \approx 1.67$, $z \approx 1.6$, and $\eta \approx -0.4$, in comparison with the accepted values [7] 1.73, 1.58 and -0.31 respectively.

These are quite close, considering that this is only a simple one-loop estimate. In fact, this agreement confirms the broad universality of the DP class. The usual field theory approach to DP involves either an ϵ -expansion about $d = 4$ transverse dimensions, or a loop expansion of a theory with only cubic couplings and no bare quartic coupling. Carried to sufficiently high order and resummed, these give values of the exponents in fair agreement

with those of simulations and enumerations. By contrast, our expansion is in the cubic couplings at *infinite* value of the repulsive quartic coupling. In principle this could lead to a different universality class. However, the good agreement in the exponents suggests that this is not the case and that, as has been observed elsewhere, DP universality is particularly strong.

We were originally motivated to study this problem in an attempt to resolve some of the confusion surrounding the behaviour of the q -state Potts model with random bonds. A rigorous result of Aizenman and Wehr [8] shows that in this case the first-order transition should be smoothed for all q , in the sense that there should be no latent heat. This suggests, by analogy with other known systems, that the interfacial tension should also vanish at the critical point. Initial Monte Carlo studies for this model for $q = 8$ found the transition indeed to be continuous, with both magnetic and thermal critical exponents consistent with the Ising values $\beta = \frac{1}{8}$ and $\nu = \mu = 1$ [9]. However, more recent studies, using both Monte Carlo methods [10] and finite-size scaling of the transfer matrix [11], have found a magnetic exponent significantly different from this value, and, moreover, continuously varying with q , although the thermal exponent remains numerically consistent with $\nu = 1$.

An analysis of branching Potts interfaces on a hierarchical lattice, in both pure and random models, has been made by Kardar, Stella, Sartoni and Derida [1]. For the pure system they were able to write down exact recursion relations for the interfacial partition function. However, on this lattice, the transition in the pure system is continuous for all finite q , thus casting doubt on the reliability of such a model for studying the effect of quenched impurities on first-order transitions. Nevertheless, this model is sufficiently simple that the problem is tractable. These authors found that the critical behaviour of the random interface is controlled by a zero-temperature RG fixed point at which the branching weight is marginally irrelevant. This implies that the critical exponents should be independent of q . These authors suggested

that a similar mechanism may operate on a regular two-dimensional lattice, consistent with the original Monte Carlo results of Chen et al. [9], but not with the more recent work [10, 11].

It would be very interesting to analyse directly the effect of impurities on Potts interfaces in two dimensions. In studying this in the SOS approximation one immediately meets a difficulty. In the bulk Potts model, quenched bond randomness is marginally irrelevant for $q = 2$ and relevant for $q > 2$. The examples of hierarchical lattices in Ref. [1] also have this property. However, in the SOS approximation, the randomness turns out to be strongly relevant even in the Ising case. Indeed, this is the well-studied problem of a directed polymer in a random medium, which may also be mapped to the KPZ problem in one dimension [12]. Therefore we conclude that the SOS approximation to the interface does not capture the correct physics of the random Ising model, nor, by extension, that of the Potts model near $q = 2$. It is of course, interesting to enquire what is the effect of branching on directed polymers in a random medium, even though this has no apparent relevance to the random Potts model. Curiously, we find that it is again marginal, as in the models studied by Kardar et al. [1].

There is long history of study of Potts interfaces, begun in the 1980s. Much of this concerned the wetting properties of the disordered phase for $q > q_c$ [13], but there were also some attempts to understand the critical properties for $q < q_c$ through the branching structure of the interface [14]. It would appear that such analytic calculations took into account only the ‘self-energy’ diagrams of our analysis, not the renormalisation of the branching vertex. As we show, the latter is essential in deriving the existence of a critical value of q from this point of view. In addition, our RG analysis actually takes account of a potentially infinite number of nested branchings and self-energy bubbles.

The layout of this paper is as follows. In the next section we define more carefully the diagrammatic expansion we use to model branching Potts inter-

faces in the SOS approximation, and argue that it may be derived from a local Lagrangian field theory. Then we discuss the renormalisation of this theory, calculating explicitly to one loop order and deriving the results given above. Finally in Sec. 3 we discuss the random case and make further comments on the problem.

2 Field theory of branching interfaces

As discussed in the Introduction, the fact that configurations of branching Potts interfaces resemble Feynman diagrams for BARWs and DP does not imply that they are in direct correspondence. As is well known, a bare Feynman propagator may be interpreted as the partition function $Z(t, x)$ for sum of directed paths from $(0, 0)$ to (t, x) weighted by a factor z_0^{length} : on the diagonal square lattice, for example, such a partition function satisfies

$$Z(t, x) = z_0 (Z(t-1, x+1) + Z(t-1, x-1)) + \delta_{t,0} \delta_{x,0} \quad (2)$$

so that, in the infrared limit of interest, its Laplace-Fourier transform has the form

$$G_0(s, k) = (s + D_0 k^2 + \sigma_0)^{-1} \quad (3)$$

where, in this example, $\sigma_0 = 2z_0 - 1$. However, a sum over all such paths in a diagram such as Fig. 1 would include configurations in which paths corresponding to different propagators cross each other, and this is unphysical in the case of interfaces.

This may be taken into account by incorporating a repulsion between neighbouring walks, as follows. Ignoring for the time being the q -dependent weight factors, the partition function for the interface pinned as described in the introduction corresponds to the correlation function

$$G(L, 0) = \int \mathcal{D}\bar{\phi} \mathcal{D}\phi \phi(L, 0) \bar{\phi}(0, 0) e^{-S} \quad (4)$$

where the action S has the general form familiar from directed percolation [3]

$$S = \int dt dx [\bar{\phi} \partial_t \phi + D_0 (\partial_x \bar{\phi}) (\partial_x \phi) + \sigma_0 \bar{\phi} \phi - \frac{1}{2} u_0 (\bar{\phi}^2 \phi + \bar{\phi} \phi^2) + \frac{1}{4} \lambda_0 \bar{\phi}^2 \phi^2] \quad (5)$$

In the absence of branching, $G = G_0$ as defined above. With branching, we still expect that $G(L, 0) \sim e^{-\sigma L}$ as $L \rightarrow \infty$, and so we may interpret the renormalised ‘mass’ σ as the interfacial tension. The term proportional to $\lambda_0 > 0$ provides an effective repulsion between different pieces of the interface. Strictly we should take the limit of infinite λ_0 , but, as with other self-repelling walk problems, finite repulsion is expected to be in the same universality class. In fact, our explicit calculations will be for infinite coupling, which then requires special treatment, described later.

The diagrams generated by (5) do not have the correct weights to represent branching Potts interfaces. While for a given diagram with a small number of loops it is relatively easy to compute this weight, which is simply the number of colourings of the internal loops consistent with neighbouring regions being coloured differently, it is not clear how to give a rule for doing this to all orders in such a way that the theory is obviously multiplicatively renormalisable. This is because the colouring problem appears to involve satisfying global constraints which are not simply given by simple rules at the local level of the vertices. However, it is possible to generalise (5) in such a way that these weights will be generated from a local Lagrangian and thus, most likely, correspond to a renormalisable field theory. Observe that each segment of interface separates two different states of the Potts model, say a and b . The corresponding propagator may therefore be represented by a ‘fat’ line carrying labels a and b , as shown in Fig. 2. Such a fat line represents the propagator $\langle \phi_a^b(x, t) \bar{\phi}_a^b(0, 0) \rangle$ of a *matrix*-valued field ϕ_a^b . Since $a \neq b$, this transforms according to a generically irreducible representation of the permutation group of q objects. The branching vertices are now represented

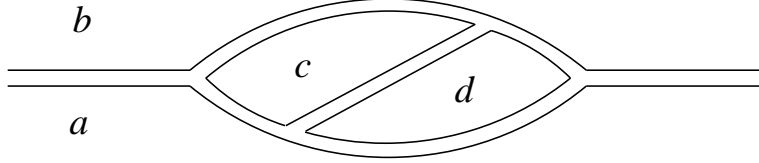


Figure 2: Fat graph corresponding to Fig. 1.

by interactions of the form

$$-u_0 \bar{\phi}_a^b \phi_a^c \phi_c^b + \text{h.c.}, \quad (6)$$

and, in any given loop graph, the internal indices are summed subject to all the constraints. Similarly, the repulsion between neighbouring interfaces may be modelled by an interaction of the form

$$\lambda_0 \bar{\phi}_b^a \bar{\phi}_c^b \phi_b^a \phi_c^b \quad (7)$$

This construction still, however, includes unphysical non-planar graphs. These may be removed by a well-known trick. Add additional $O(N)$ indices (i, j) to the fields, where $1 \leq i, j \leq N$, and generalise (6) to

$$-u_0 \bar{\phi}_{a,i}^{b,j} \phi_{a,i}^{c,k} \phi_{c,k}^{b,j} + \text{h.c.}, \quad (8)$$

where the internal $O(N)$ indices are summed freely. On taking the limit $N \rightarrow \infty$, $u_0 \rightarrow 0$, with $u_0^2 N$ fixed, only the planar diagrams survive.

Having argued that the configurations of branching interfaces can be viewed as the Feynman diagrams of a local Lagrangian field theory, we shall in fact make no further reference to this. Since we shall be concerned only with low order calculations, the relevant diagrams and their weight factors will be straightforward to write down by inspection.

2.1 Fermionic representation and perturbative calculation

It is possible to develop and renormalise the perturbative expansion of G simultaneously in powers of both the branching u_0 and the repulsion λ_0 .

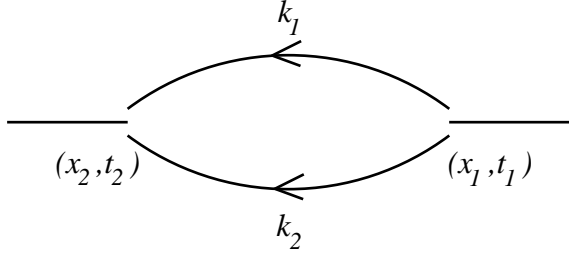


Figure 3: Point-split version of the one-loop contribution to $\Gamma^{(1,1)}$.

In the absence of branching there is a non-trivial fixed point in transverse dimension $d < 2$, which corresponds to the limit of infinite bare parameter λ_0 . However, the cubic branching term becomes marginal at $d = 4$, just as in DP, and so it is strongly relevant for $d < 2$. It is of course possible to consider the truncated loop expansion in both λ_0 and u_0 , but there is no reason for it to give sensible results for the physical case $d = 1$. Indeed, the one-loop calculations we have performed for this case appear to lead to no relevant fixed point in $d = 1$ describing the interface problem.

Instead, we shall adopt a different strategy which is appropriate only in the physical number of transverse dimensions $d = 1$. In that case we may observe that, in between branchings, the interfaces behave like the world lines of non-relativistic bosons with infinitely strong hard core repulsion. In one dimension, these are equivalent to free fermions. The interfaces thus propagate as fermions in between branching events. However, since no more than one particle may then occupy the same site, it now becomes necessary to smear the branching process so that particles are shifted to neighbouring sites of the lattice. Of course this should not affect the universal behaviour.

Consider, for example, the simple bubble diagram shown in Fig. 3, evaluated in real space initially. The pair of walks in the loop begin and end at (t_1, x_1) and (t_2, x_2) respectively, but we separate them so these become $(t_1, x_1 \pm a)$ and $(t_2, x_2 \pm a)$, where a is of the order of the lattice spacing. The propagator for such a pair of walks which avoid each other is given by the

method of images as

$$\begin{aligned} & G_0(t_1, x_1 + a; t_2, x_2 + a) G_0(t_1, x_1 - a; t_2, x_2 - a) \\ & - G_0(t_1, x_1 + a; t_2, x_2 - a) G_0(t_1, x_1 - a; t_2, x_2 + a), \end{aligned} \quad (9)$$

or, in terms of Fourier transforms

$$\int \frac{dk_1 dk_2}{s + D_0(k_1^2 + k_2^2) + 2\sigma_0} \left(e^{ik_1(x_1 - x_2)} e^{ik_2(x_1 - x_2)} - e^{ik_1(x_1 - x_2 + 2a)} e^{ik_2(x_1 - x_2 - 2a)} \right) \quad (10)$$

Expanding the last factor to lowest non-vanishing order then gives $\frac{1}{2}(2a)^2(k_1 - k_2)^2$. Each vertex thus acquires a wave number dependence $\propto (k_1 - k_2)$. The factor $\frac{1}{2}$ is the usual symmetry factor for this diagram. We may absorb the factors of $2a$ into the branching rate u_0 to give a new effective coupling constant $\tilde{u}_0 \equiv 2au_0$. Notice this will have a different canonical dimension. The calculation of Fig. 3 is therefore equivalent to that in a theory with Grassmann fields $c(x, t)$, $\bar{c}(x, t)$, with the same propagator but with interactions

$$\frac{1}{2}\tilde{u}_0(\bar{c}\partial_x \bar{c}c + \bar{c}c\partial_x c) \quad (11)$$

Note that this is the lowest order non-vanishing cubic interaction consistent with the rules $c^2 = \bar{c}^2 = 0$.

In general, this fermionic correspondence extends to higher order diagrams. There is one important complication however. The interfaces behave as free fermions only in between branching or coagulation events, wherever these may be, which may involve other particles. An example is the one-loop renormalisation of the vertex, shown in Fig. 4. In this case the interface passing through the point u in the figure must avoid the branching at point y , which occurs at the same ‘time’ t' . While the fermionic formulation ensures that the interfaces immediately to the right and left of t' avoid each other, if we were to take the fermionic lagrangian literally we would integrate over all values of the intermediate coordinate u . This would include the region $u < y$

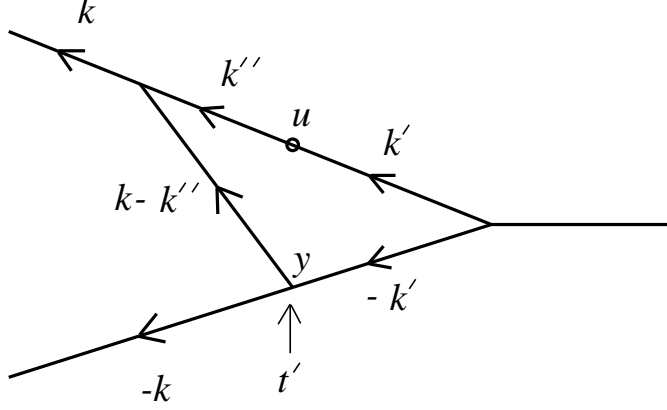


Figure 4: One-loop renormalisation of the coupling constant. In this diagram, the physical region is restricted to $u > y$.

which is unphysical and where, in fact, the amplitude has the wrong sign. Thus, in using the fermionic formulation, we must be careful to integrate over only the physical part of the phase space, where the ‘particles’ retain their original ordering. This is in contrast to what would be done in a fully fermionic quantum field theory, where it would be necessary to integrate over all of phase space so as to respect the Fermi statistics of the wave functions.

The restriction to $u > y$ makes the calculation of the Feynman integral considerably more complicated. However, we shall argue that as long as we are interested only in the renormalisation of the infrared limit of the vertex, we need simply to multiply the unrestricted diagram by a factor $\frac{1}{2}$. Similar rules are expected to hold for more complicated graphs.

We are now in a position to summarise the Feynman rules:

1. Propagator $(s + D_0 k^2 + \sigma_0)^{-1}$;
2. Vertices $\pm i(k_1 - k_2)\tilde{u}_0$, where k_1 and k_2 are the wave-numbers of the pair of outgoing (incoming) fermions;
3. Usual symmetry factors;

4. q -dependent weights corresponding to the number of colourings of the diagram. There will also be factors of (-1) from fermion Wick contractions. These in fact cancel minus signs from the vertices. The overall sign of a diagram is, in any case, always positive for sufficiently large positive q .
5. Factors due to the restriction of phase space, as discussed above.

2.2 One-loop renormalisation

In the usual way, we may define the one-particle irreducible vertex functions $\Gamma^{(1,1)}$ and $\Gamma^{(2,1)}$ for this theory and compute the renormalisation of the fields, the diffusion constant and the coupling constant [15]. We adopt the scheme of renormalising in the massive theory ($\sigma > 0$) at zero external frequency and wave number, which makes the integrals somewhat simpler [16]. From Fig. 3 we have

$$\begin{aligned} \Gamma^{(1,1)}(s, k) = & s + D_0 k^2 + \sigma_0 \\ & - \frac{1}{2}(q-2)\tilde{u}_0^2 \int \frac{dk'}{2\pi} \frac{(2k')^2}{s + D_0(k' + \frac{1}{2}k)^2 + D_0(k' - \frac{1}{2}k)^2 + 2\sigma_0} + O(\tilde{u}_0^4) \end{aligned}$$

Imposing the normalisation condition

$$(\partial/\partial s)\Gamma_R^{(1,1)}(0,0) = 1 \quad (12)$$

with $\Gamma_R^{(1,1)} = Z_c \Gamma^{(1,1)}$, yields the field renormalisation

$$Z_c = 1 - \frac{1}{2}(q-2)\tilde{u}_0^2 \int \frac{dk'}{2\pi} \frac{4k'^2}{(2D_0k'^2 + 2\sigma_0)^2} = 1 - \frac{1}{8}(q-2)g_0^2 + O(g_0^4), \quad (13)$$

where $g_0^2 = \tilde{u}_0^2/D_0^{3/2}\sigma_0^{1/2}$. Similarly, we find for the renormalised diffusion constant

$$D_R \equiv \frac{\partial}{\partial k^2} \Gamma_R^{(1,1)}(0,0) = Z_c D_0 \left(1 + \frac{1}{16}(q-2)g_0^2 + O(g_0^4) \right) \quad (14)$$

Next, we have the vertex renormalisation, given at one loop by two diagrams like Fig. 4. The calculation of this diagram is complicated by the restriction $u > y$. For this reason it is not true that the wave number flowing through the point u is simply conserved. Instead, the integration over u yields

$$\int_y^\infty e^{-ik'u+ik''u-\epsilon u} du = \frac{-ie^{-ik'y+ik''y}}{k' - k'' - i\epsilon}, \quad (15)$$

where we have inserted the factor $e^{-\epsilon u}$, with $\epsilon \rightarrow 0+$, to ensure convergence. Now use the identity $(x-i\epsilon)^{-1} = P(1/x) + i\pi\delta(x)$, and observe that the whole diagram is invariant under simultaneous complex conjugation and sending $k \rightarrow -k$. The above decomposition will therefore lead to two terms with opposite parity under $k \rightarrow -k$. The second term is precisely one half of what would be obtained if we had ignored the restriction $u > y$, and, as we see explicitly below, it has the form ik times a real function of k^2 . By contrast, the first term must therefore yield a real function of k^2 only. Since we are interested in the contribution $\propto ik$ in $\Gamma^{(2,1)}(k, -k, 0)$ in the infrared limit $k \rightarrow 0$, it follows that we need retain only the second term, which is one half of what we would have found imposing wave number conservation $k' = k''$.

The result is therefore

$$\Gamma^{(2,1)} = 2ik\tilde{u}_0 + \frac{1}{2}2(q-3)\tilde{u}_0^3 \int \frac{dk'}{2\pi} \frac{2ik' i(k-k') i(2k-k')}{(2D_0k'^2 + 2\sigma_0)(D_0k'^2 + D_0(k-k')^2 + 2\sigma_0)} \quad (16)$$

where the other important feature to note is the factor of $q-3$ which comes from the fact that there are three different colours on the boundary, and the internal colour must differ from each of these. In this integral we need take only the term $O(k)$ as $k \rightarrow 0$. After some straightforward quadratures we find

$$\Gamma^{(2,1)} = 2ik\tilde{u}_0 \left(1 + \frac{7}{32}(q-3)g_0^2 + O(g_0^4)\right) + O(k^3) \quad (17)$$

From these expressions we may find the dimensionless renormalised cou-

pling

$$g \equiv \frac{(2ik)^{-1}\Gamma^{(2,1)}Z_c^{3/2}}{\sigma^{1/4}D_R^{3/4}} = \frac{\tilde{u}_0}{\sigma^{1/4}D_0^{3/4}} \left[1 + \left(\frac{7}{32}(q-3) - \frac{9}{64}(q-2) \right) g_0^2 + \dots \right] \quad (18)$$

Notice that g_0 depends on the bare mass σ_0 , but, to the order indicated, we may replace this by σ . We then finally have the beta-function

$$\beta(g) \equiv \sigma \frac{\partial g}{\partial \sigma} \Big|_{\tilde{u}_0} = -\frac{1}{4}g - \frac{1}{2} \left(\frac{7}{32}(q-3) - \frac{9}{64}(q-2) \right) g^3 + \dots \quad (19)$$

This is the main result (1) quoted in the introduction. We have left the second term in this form to illustrate how the critical value $q_c \approx q_0 = \frac{24}{5}$ (in this approximation) emerges from the vertex renormalisation, proportional to $q-3$, and that of the propagator, proportional to $q-2$. In particular, one may see how the latter increases q_c above the naively expected value of three. This also illustrates why the hierarchical lattices of Ref. [1] give a different result, for in these cases there are only even factors of $q-2$, and no vertex corrections.

2.3 Calculation of the Widom exponent

As discussed in the introduction, the renormalised interfacial tension σ is supposed to vanish close to a continuous transition as $(\sigma_0 - \sigma_{0c})^\mu$. In the language of directed percolation, μ may therefore be identified with the exponent ν_{\parallel} (or, in the language of dynamic critical behaviour, $z\nu$.) The method of estimating its value using the field-theoretic renormalisation group is standard [15]. Define the vertex $\Gamma^{(1,1)}$ with an insertion of $\bar{c}c$ as

$$\begin{aligned} \Gamma^{(1,1;1)} \equiv (\partial/\partial\sigma_0)\Gamma^{(1,1)}(0,0) &= 1 + (q-2)\tilde{u}_0^2 \int \frac{dk'}{2\pi} \frac{(2k')^2}{(2D_0k'^2 + 2\sigma_0)^2} + \dots \\ &= 1 + \frac{1}{4}(q-2)g_0^2 + \dots \end{aligned} \quad (20)$$

In turn, this gives the combination of renormalisation constants $Z_c^{-1}Z_{\bar{c}c}$, if we normalise so that $\Gamma_R^{(1,1;1)} = 1$ at zero frequency and wave number. Thus

we find that

$$\gamma_{\bar{c}c} \equiv \sigma \frac{\partial \ln Z_{\bar{c}c}}{\partial \sigma} = -\frac{1}{16}(q-2)g^2 + \dots \quad (21)$$

In the standard way, at the fixed point $\gamma_{\bar{c}c}^* = 1 - \nu_{\parallel}^{-1}$, which yields, in this one-loop approximation

$$\mu = \nu_{\parallel} \approx \frac{24 - 5q}{20 - 3q} \quad (22)$$

Obviously this is only a rather crude estimate and it is difficult to gauge its accuracy. However it does exhibit the correct trend as q varies close to 2, and in fact for $q = 3$ gives the remarkably good estimate $\mu \approx \frac{9}{11}$ to be compared with the exact value $\frac{5}{6}$.

In the context of critical dynamics or DP it is also possible to define two other independent critical exponents, namely the dynamical exponent z and η_{\parallel} , defined by the scaling law

$$G(t, k) \sim t^{-\eta_{\parallel}} F(tk^z) \quad (23)$$

for the Fourier transform with respect to x of $G(x, t)$, evaluated at criticality. For simple non-branching directed paths we have $z = 2$ and $\eta_{\parallel} = 0$. The latter result agrees with that expected for an Ising interface whose ends are pinned at the points $(0, 0)$ and (L, x) , after the partition function, which corresponds to $G(L, x)$, is integrated over x . This is because the pinning sites correspond to insertions of disorder operators at the boundary points $(0, 0)$ and (x, L) : these are expected to scale as L^{-2x_s} , where $x_s = \frac{1}{2}$ is the boundary scaling dimension of such an operator, equal by duality to the boundary dimension of the magnetisation. The additional integration over x then provides another factor of L , giving $\eta_{\parallel} = 2x_s - 1 = 0$. On the other hand, the result $z = 2$ may not simply be compared with bulk Ising exponents, since it refers to the SOS picture of the interface, which neglects overhangs and bubbles of the opposite phase and therefore cannot be directly compared with, for example, the width of the magnetisation profile which is expected to scale as L^1 at the critical point.

Within the field-theoretic formulation of the interface for $q > 2$, these two exponents are related, in the standard way, to the fixed point values of the renormalisation group functions $\gamma_c \equiv \sigma(\partial/\partial\sigma) \ln Z_c$, and $\gamma_D \equiv 2\sigma(\partial/\partial\sigma) \ln Z_D$, where $D_R = Z_D D_0$. Explicitly, $\eta_{\parallel} = \gamma_c^*$ and $z = 2 + \gamma_D^*$. In the one-loop approximation, this gives

$$\eta_{\parallel} \approx \frac{1}{16}(q-2)g^2 \approx \nu_{\parallel}^{-1} - 1 \quad (24)$$

$$z \approx 2 + \frac{1}{16}(q-2)g^2 \approx \nu_{\parallel}^{-1} + 1 \quad (25)$$

For $q = 3$, for example, this leads to the estimate $\eta_{\parallel} \approx \frac{2}{9}$, to be compared with the exact value $2x_s - 1 = \frac{1}{3}$ [17]. For z we find $\approx \frac{20}{9}$, consistent with this measure of the interfacial width becoming larger as the interface becomes more branched.

2.4 The directed percolation limit

As discussed in the introduction, the configurations of a branching Potts interface resemble the diagrams of the field theory of directed percolation (DP) [3], but, in general, the weights are different, since in DP these are simply given by a factor (-1) for each closed loop. This weighting may be recovered, however, by taking the formal limit $q \rightarrow -\infty$ with $\tilde{u}_0^2(-q)$ fixed. if we take this limit in Eqs. (22,24,25) we find the estimates $\nu_{\parallel} \approx 1.67$, $z \approx 1.6$ and $\eta_{\parallel} \approx -0.4$ (the latter exponent gives the rate of growth of the average number of infected sites $t^{-\eta_{\parallel}}$ from a single seed). As discussed in the introduction, these agree remarkably well with currently accepted values from enumerations and Monte Carlo methods. This success at the one-loop approximation may be attributed to the inclusion of the infinitely strong repulsion between neighbouring walks. As discussed above, this implies that the effective fermionic interaction vanishes at zero wave number, and as a consequence the canonical dimension of the effective coupling is reduced from $[u_0] = k^{3/2}$ to $[\tilde{u}_0] = k^{1/2}$. Equivalently, one may say that the upper critical

dimension of the field theory is reduced from $d = 4$ transverse dimensions to $d = 2$ (although, strictly speaking, the fermionic theory makes sense only for $d = 1$). Thus we expect the loop expansion to give more accurate results when truncated at a low level. It would be interesting to extend this analysis to two loops.

However, the fact that the theory when expanded about infinitely large quartic coupling λ_0 gives exponents which appear to agree with those obtained from the theory with zero bare quartic coupling is non-trivial, and provides further evidence for the robustness of the DP universality class.

3 Further remarks

3.1 Effect of randomness

As discussed in the introduction, we were originally motivated to study this problem by the paper of Kardar et al. [1] on the effect of bond disorder on the branching Potts interface. Their model, on certain types of hierarchical lattice, is unphysical in that exhibits a continuous transition for all q in the pure case, but it does have one important feature in common with Potts models on regular 2d lattices: namely that the randomness is relevant for $q > 2$, and marginally irrelevant for the Ising case $q = 2$. This feature persists to the interfacial model since it is exact on the hierarchical lattice. This enabled Kardar et al. [1] to draw exact conclusions for their model, in particular that the branching is marginally irrelevant. Whether these conclusions also hold for the random Potts models on regular lattices is of course debatable, and in the light of recent evidence [10, 11] probably incorrect.

It would be very useful if randomness could be incorporated into a more realistic yet solvable interface model. Unfortunately the SOS approximation, while it captures most of the features of the pure model, fails to do so in the random case. This may already be seen for $q = 2$. In that case, the random

bonds act as randomly positioned pinning centres for the interface. This is equivalent to the well-studied problem of a directed polymer in a random medium, which may be mapped onto the KPZ equation [12]. Unlike the case of the bulk random bond Ising model, the randomness in this SOS model is strongly relevant, changing, in particular, the dynamic exponent z from two to the KPZ value of $\frac{3}{2}$. This implies that the SOS model is not a useful one in which to study the effects of randomness. Indeed, it shows that the contributions of overhangs and bubbles of the opposite phase do not simply cancel in the random case, at least near bulk criticality.

Nevertheless, it is interesting to investigate whether branching of such randomly pinned directed polymers is relevant. A simple scaling argument in fact indicates that it should be marginal, as in the case studied by Kardar et al [1]. If we use the fermionic lagrangian of (11) we may extract the following dimensions in terms of transverse wave number k and ‘frequency’ ω : $[\bar{c}c] = k$, implying that $[\bar{c}] = [c] = k^{1/2}$. Thus $[\tilde{u}_0] \cdot k \cdot k^{3/2} = \omega k = k^{1+z}$. Using the KPZ value $z = \frac{3}{2}$ we see that \tilde{u}_0 is dimensionless, and therefore marginal. Unfortunately it is much more difficult to determine whether or not it is marginally irrelevant. It would be interesting to understand whether the marginality of branching is a stable property of pinned directed interfaces, or whether the similarity between the hierarchical and euclidean 2d cases is coincidental.

3.2 The effect of the disordered phase

Although the one-loop approximation to the renormalisation group equations for the SOS branching Potts interface captures much of the physics of the pure Potts model, even quantitatively, there is one important feature lacking, which is the description of the behaviour as $q \rightarrow q_c$. If we integrate (1) for $q > q_c$ to calculate the renormalised interfacial tension at which g diverges (which we interpreted as the first-order transition) we find $\sigma \sim \sigma_0 e^{-1/2b(q-q_c)}$.

This is to be compared with the exactly known behaviour $\sim e^{-\text{const}/(q-q_c)^{1/2}}$ [5]. Similarly, our model suggests that, as $q \rightarrow q_c-$, $\mu \rightarrow 0$, while in fact it retains a finite value at $q = q_c = 4$, jumping discontinuously to an (effective) value of zero for $q > 4$.

These two features, which are missing in our approach, are qualitatively accounted for in the picture of the fixed point structure which emerges from approximate real space RG calculations which incorporate annealed vacancies as additional degrees of freedom [6]. These suggest that, for $q < q_c$, there are two fixed points, corresponding respectively to critical and tricritical behaviour, which merge in a parabolic fashion at $q = q_c$, thus giving rise to a marginal operator at this point. In fact, this qualitative picture has been shown to lead to all sorts of generalised scaling relations near $q = 4$ which have been verified by exact results and by Monte Carlo simulations [18, 19].

It is clear what is missing in our model for $q \geq q_c$. In this case, at the bulk critical point, there is coexistence not only between the q ordered phases, but also with the disordered phase. Thus, in the interfacial model, we should take into account not only interfaces between different ordered phases, but also those between any ordered phase and the disordered phase. Both types of renormalised interfacial tension are expected to vanish at bulk criticality. Indeed, it has been found both numerically and analytically [13] that the disordered phase wets the interface between two ordered phases at $T = T_c$ for $q > q_c$. The resulting SOS model has, potentially, a very rich structure. Each type of interface will have its own bare ‘mass’ or interfacial tension, and, in principle, its own diffusion constant. In addition, there are now three different types of branching process which reflect themselves in three *a priori* independent couplings to be renormalised. It remains to be seen whether the known behaviour for $q > q_c$ may be fitted into this picture, but, given that vacancies may be regarded as microscopic regions of disordered phase, there is every expectation that this should be the case. It would also be interesting to apply the ideas of this paper to other more complicated models (e.g. the

Ashkin-Teller model) with more than one type of interface.

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